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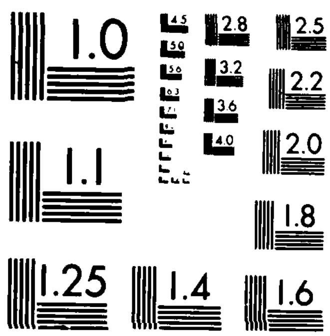
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ON THE 0,1 FACETS OF THE SET COVERING POLYTOPE

April 1986

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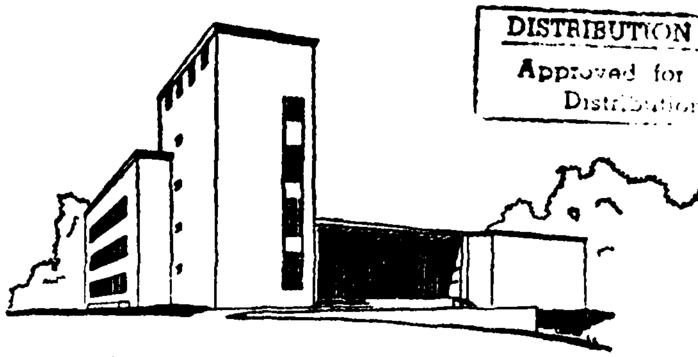

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ABSTRACT

In this paper, we consider inequalities of ~~the~~ form $\cancel{\sum \alpha_j x_j \geq \beta}$, ~~where~~ $\cancel{\alpha_j}$ equals 0 or 1, and β is a positive integer. We give necessary and sufficient conditions for such inequalities to define facets of the set covering polytope associated to a 0,1 constraint matrix A. These conditions are in terms of critical edges and critical cutsets defined in the bipartite incidence graph associated to A, and are very much in the spirit of the work of Balas and Zemel on the set packing problem where similar notions were defined in the intersection graph of A. Furthermore, we give a polynomial characterization of a class of 0,1 facets defined from chorded cycles induced in the bipartite incidence graph. This characterization also yields all the 0,1 liftings of odd-hole inequalities for the simple plant location polytope.

1. INTRODUCTION

Let A be an m×n matrix with 0,1 elements. The convex hull of the solutions to

$$\left.\begin{aligned} &Ax \leq 1 \\ &x_j \in \{0,1\} \quad j = 1,\ldots,n \end{aligned}\right\} \tag{1}$$

has been widely studied in the 1970's see e.g. Fulkerson (1971), Padberg (1973), Nemhauser and Trotter (1974), Trotter (1975), Chvatal (1975), Balas and Padberg (1976), Wolsey (1976), Balas and Zemel (1977). This polytope is known as the *set packing polytope* and we denote it by P(A).

The *set covering polytope* C(A) is the convex hull of the solutions to

$$\left.\begin{aligned} &Ax \geq 1 \\ &x_j \in \{0,1\} \quad j = 1,\ldots,n. \end{aligned}\right\} \tag{2}$$

The facial structure of C(A) has received considerable attention only recently, e.g. Balas and Ng (1984, 1986), Conforti, Corneil and Majoub (1984), Sassano (1985). Related work involves the use of both the polytope C(A) and an objective function to derive cutting planes as proposed by Balas (1979, 1980), Balas and Ho (1980).

This time lag between the study of P(A) and C(A) may appear surpising considering the practical importance of the set covering problem. A possible explanation

is that several key concepts for the study of the set packing polytope P(A) did not seem to be transferable to the covering problem. In this paper, we explore such possibilities. We concentrate on valid inequalities of the form

$$\sum_{j \in S} x_j \leq \alpha \qquad (3)$$

for the set packing problem and

$$\sum_{j \in S} x_j \geq \beta \qquad (4)$$

for the set covering problem, where $S \subseteq \{1,...,n\}$ and α, β are positive integers. In other words, we consider linear inequalities where the coefficients of the variables x_j, for $j = 1,...,n$, are equal to 0 or to 1.

The notions of critical edges and critical cutsets that were introduced by Chvatal (1975) and Balas and Zemel (1977) for the set packing problem can also be defined - with respect to another graph - for the set covering problem. Chvatal used critical edges to give a sufficient condition for the inequality (3) to define a facet of P(A). Sassano (1985) showed that the same condition holds for the set covering problem, provided the new framework is used. In section 2,3 and 4 we parallel the results of Balas and Zemel (1977). In particular we show in section 2 that the sufficient condition of Sassano is not necessary. In section 3 we give a necessary condition in terms of critical cutsets but we show that it is not sufficient. Finally, in section 4, we give a necessary and sufficient condition for (4) to define a facet of C(A).

In section 5, we investigate set covering problems

that have a given inequality (4) as a facet. In particular we consider β-maximal 0,1 matrices A,where maximality refers to the property that changing any element of A from 0 to 1 would render the inequality (4) invalid for C(A). For this class of matrices the sufficient condition of Sassano is also necessary. This is to be related to the 0,1 facets of the simple plant location polytope which are generated from β-maximal matrices, see Cho, Padberg and Rao (1983). For these facets, it has also been proved that the condition of Chvatal is necessary and sufficient, see Cornuejols and Thizy (1982).

Square matrices of odd order with exactly two ones per row and per column are at the hart of the polyhedral theory of set packing and set covering problems. Define these matrices as *odd holes*. If A does not contain an odd hole as a submatrix, then $P(A) = \{x: Ax \leq 1,\ 0 \leq x \leq 1\}$ and $C(A) = \{x: Ax \geq 1,\ 0 \leq x \leq 1\}$. When these equalities do not hold, the facets associated with odd holes are of particular importance. In section 6, we give a polynomial characterization of all the β-maximal matrices arising from odd holes.

2. CRITICAL EDGES

For the set packing problem, a useful notion has been that of the *intersection graph* of a 0,1 matrix A. It is defined as having a node for each column of A and an edge for each pair of nonorthogonal columns. It is well known and easy to check that the feasible solutions of the set packing problem (1) are in one-to-one correspondence with the node packings of the intersection graph of A.

Given a graph G, let $\alpha(G)$ be the maximum cardinality of a node packing in G. An edge of G is said to be *critical* if $\alpha(G \setminus e) > \alpha(G)$, where $G \setminus e$ denotes the graph obtained from G by deleting the edge e.

THEOREM 1 (Chvatal 1975). *Let* $G = (V,E)$ *be the intersection graph of a* 0,1 *matrix* A. *Let* $E^* \subseteq E$ *be the set of the critical edges of* G. *If* $G^* = (V,E^*)$ *is connected, then*

$$\sum_{j=1}^{n} x_j \leq \alpha(G)$$

defines a facet of the set packing polytope P(A).

For the set covering problem, the intersection graph of A appears to be less appropriate for the study of C(A). Rather, we define the *bipartite incidence graph* of a 0,1 matrix A as follows. The graph $B = (V,U,E)$ has a node $i \in U$ for each row of A, a node $j \in V$ for each column of A and an edge between nodes $i \in U$ and $j \in V$ if and only if $a_{ij} = 1$ in the matrix A. Consider the set $T \subseteq U$. A set $S \subseteq V$ is called a *cover of* T if every $i \in T$ is adjacent to at least one node of S. The feasible solutions

of the set covering problem (2) are in one-to-one correspondence with the covers of U.

Given the bipartite graph $B = (V,U,E)$ and a set $T \subseteq U$, let $\beta(T)$ be the minimum cardinality of a cover of T. Moreover denote by $G^* = (V,E^*)$ the *critical graph* associated to B having node set V and edge set (*critical edges*) defined as:

$$E^* = \{(v_i,v_j) \mid \beta(U \setminus U_{ij}) < \beta(U)\}$$

where $U_{ij} \subseteq U$ is the set of the common neighbors of v_i and v_j. We assume that $\beta(\emptyset) = 0$.

THEOREM 2 (Sassano 1985). *Let* $B = (V,U,E)$ *be the bipartite incidence graph of a* 0,1 *matrix* A. *Let* $G^* = (V,E^*)$ *be the associated critical graph. If* G^* *is connected, then*

$$\sum_{j=1}^{n} x_j \geq \beta(U)$$

defines a facet of the set covering polytope C(A).

Observe that, alternately, we denote by $\beta(A)$ the covering number of the 0,1 matrix A. In other words $\beta(A) \equiv \beta(U)$.

For the set packing polytope, the sufficient condition given in Theorem 1 is not necessary as was pointed out by Balas and Zemel (1977). A similar fact can be proved for the set covering polytope and the sufficient condition of Theorem 2.

THEOREM 3. *Let* $B = (V,U,E)$ *be the bipartite incidence graph of a* 0,1 *matrix* A *and assume that the inequality:*

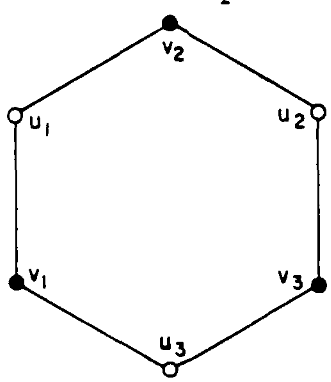
$$\sum_{j=1}^{n} x_j \geq \beta(A)$$

induces a facet of C(A).
Then the critical graph $G^* = (V,E^*)$ *is connected for some choices of* A *and disconnected for other choices.*

PROOF. We give an example where G^* is connected and one where G^* is disconnected.
First consider the matrix:

$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \end{array}$$

and the associated incidence bipartite graph

B: 

Any edge (v_j,v_k) is critical as $\beta(U) = 2$ and $\beta(U \setminus U_{jk})=1$ where the set U_{jk} is defined as above. Therefore G^* is the complete graph on 3 nodes and thus is connected. Of course $\sum_{j=1}^{3} x_j \geq 2$ is a facet by Theorem 2.

Now consider the following 0,1 matrix

$$
A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{array}
\begin{array}{c}
\begin{array}{ccccccccccc} 1 & 2 & 3 & 4 & 5 & & 6 & 7 & 8 & 9 & 10 \end{array} \\
\left(\begin{array}{ccccc|ccccc}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
\hline
1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1
\end{array}\right)
\end{array}
$$

And the associated incidence bipartite graph displayed in figure 1.

$B = (V,U,E)$

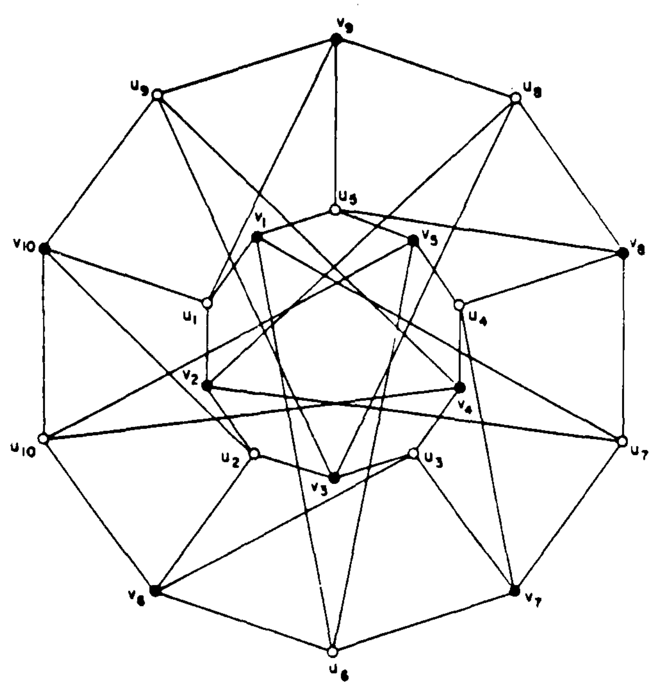


Figure 1

Note that $\beta(U) = 3$ ($|U| = 10$ and each $v_i \in V$ has degree 4

in B; so $\beta(U) \geq 3$. Furthermore the set $\{v_1, v_2, v_4\}$ is a cover of U).

Now consider the critical graph $G^* = (V, E^*)$ associated to B and, for each pair v_j, v_k of nodes of V denote, as above, by $U_{jk} \subseteq U$ the set of common neighbors to v_j and v_k. The edge (v_j, v_k) belongs to E^* if and only if $\beta(U \setminus U_{jk}) \leq 2$.

If $|U_{jk}| \leq 1$, then $|U \setminus U_{jk}| \geq 9$. But then $\beta(U \setminus U_{jk}) > 2$ as each node $v_i \in V$ has degree 4. So, in this case, (v_j, v_k) is *not* critical.

Now assume that $|U_{jk}| \geq 2$. Actually the graph B has the property that $|U_{jk}| \leq 2$. So we get $|U_{jk}| = 2$. We distinguish two cases.

(a) $1 \leq j, k \leq 5$ or $6 \leq j, k \leq 10$; $|U_{jk}| = 2$.

Then we will show that (v_j, v_k) is a critical edge.

By symmetry, we can assume w.l.o.g. that $j = 1$ and $k = 2$. Then $U_{12} = \{u_1, u_6\}$. The set $\{v_3, v_5\}$ is a cover of $U \setminus U_{12}$ and therefore $\beta(U \setminus U_{12}) = 2$.
This proves that (v_j, v_k) is critical.

(b) $1 \leq j \leq 5$ and $6 \leq k \leq 10$; $|U_{jk}| = 2$.

Then we will show that (v_j, v_k) is *not* critical.

By symmetry, only two situations can occur depending on whether $U_{jk} \cap \{u_1, \ldots, u_5\}$ is even or odd. Namely, we can assume w.l.o.g. that either $j = 1$, $k = 6$ or $j = 1$, $k = 10$. Now we study these two subcases.

(b1) $j = 1$ and $k = 6$.

In $U \setminus U_{16}$, there are five nodes to cover in the set

$\{u_1,\ldots,u_5\}$ and three in the set $\{u_6,\ldots,u_{10}\}$. Since each node $v_i \in V$ covers two nodes from each set, there is no cover of cardinality 2. So (v_j,v_k) is *not* critical.

(b2) $j = 1$ and $k = 10$.

We have $|U \setminus U_{jk}| = 8$. So, in order to cover $U \setminus U_{jk}$ with only two nodes of V, we must use two nodes of V that cover four nodes of $U \setminus U_{jk}$ each, i.e. these two nodes must be chosen from the set $\{v_3,v_4,v_7,v_8\}$. But any two of these nodes have at least one common neighbor, and therefore do not cover $U \setminus U_{jk}$. So (v_j,v_k) is not critical.

We just proved that $G^* = (V,E^*)$ is the union of two node-disjoint cycles of length five, since only case (a) gives rise to critical edges.

It remains to show that $\sum_{j=1}^{10} x_j \geq 3$ induces a facet of C(A). It is valid since $\beta(U) = 3$. In addition, the 10 solutions defined by the columns of the matrix

$$\left(\begin{array}{ccccc|ccccc}
1 & 1 & 0 & 1 & 0 & & & & & \\
0 & 1 & 1 & 0 & 1 & & & & & \\
1 & 0 & 1 & 1 & 0 & & & 0 & & \\
0 & 1 & 0 & 1 & 1 & & & & & \\
1 & 0 & 1 & 0 & 1 & & & & & \\
\hline
 & & & & & 1 & 1 & 0 & 1 & 0 \\
 & & & & & 0 & 1 & 1 & 0 & 1 \\
 & & 0 & & & 1 & 0 & 1 & 1 & 0 \\
 & & & & & 0 & 1 & 0 & 1 & 1 \\
 & & & & & 1 & 0 & 1 & 0 & 1
\end{array}\right)$$

all satisfy the constraints $Ax \geq 1$ and verify the inequality $\sum_{j=1}^{10} x_j \geq 3$ with equality. Since the above matrix is nonsingular (it is defined using a nonsingular 5×5 circulant matrix), the 10 points that it defines are affinely independent.

Thus $\sum_{j=1}^{10} x_j \geq 3$ definesa facet of $C(A)$. □

3. CRITICAL CUTSETS

A useful concept in the study of the set packing problem is that of critical cutset. A *cutset* $F = (S,\bar{S}) \subseteq E$ of a graph $G = (V,E)$ is the set of all the edges joining nodes in S with nodes in the complement set $\bar{S} = V \setminus S$. Both S and $\bar{S}$ are assumed to be nonempty.

A cutset is *critical* if $\alpha(G \setminus F) > \alpha(G)$, where $G \setminus F$ is the graph obtained from G by removing all the edges of F.

THEOREM 4 (Balas and Zemel 1977). *Let* G *be the intersection graph of a* 0,1 *matrix* A. *If* $\sum_{j=1}^{n} x_j \leq \alpha(G)$ *induces a facet of* $P(A)$, *then every cutset of* G *is critical.*

We introduce a similar notion for the set covering problem. Consider the bipartite incidence graph $B = (V,U,E)$ of a 0,1 matrix A. Let S be a nonempty proper subset of V and let $U_S \subseteq U$ be the set of all the nodes adjacent to at least one node in S and one node in $\bar{S} = V \setminus S$. The set U_S is called a *cutset* of the bipartite incidence graph B,and

in fact, B becomes disconnected if U_S and the incident edges are removed from B. The cutset U_S induced by S and $\bar{S}$ is said to be *critical* if $\beta(U \setminus U_S) < \beta(U)$.

THEOREM 5. *Let* $B = (V,U,E)$ *be the bipartite incidence graph of a* 0,1 *matrix* A. *If* $\sum_{j=1}^{n} x_j \geq \beta(U)$ *induces a facet of* C(A), *then* <u>*every*</u> *cutset of* B *is critical.*

PROOF. Assume not. Then there exists a subset $S \subseteq V$ such that $S, \bar{S} \neq \emptyset$ and $\beta(U \setminus U_S) = \beta(U)$ where U_S is the cutset defined by S. Let $U_1 \subseteq U$ be the set of nodes of B adjacent to S but not to $\bar{S}$ and $U_2 \subseteq U$ those that are adjacent to $\bar{S}$ but not to S. Then $\beta(U \setminus U_S) = \beta(U_1)+\beta(U_2)$. Thus the inequality $\sum_{j \in V} x_j \geq \beta(U)$ can be written as the sum of the two inequalities $\sum_{j \in S} x_j \geq \beta(U_1)$ and $\sum_{j \in \bar{S}} x_j \geq \beta(U_2)$. In addition, each of these inequalities is valid. Therefore $\sum_{j \in V} x_j \geq \beta(U)$ does not induce a facet of C(A). □

The necessary condition of Theorem 4 given by Balas and Zemel for the set packing problem is not sufficient as they pointed out themselves. Similarly, for the set covering problem, the necessary condition of Theorem 5 is not sufficient. This is shown next.

THEOREM 6. *Let* $B = (V,U,E)$ *be the bipartite incidence graph of a* 0,1 *matrix* A. *Assume that every cutset is critical and consider the valid inequality* $\sum_{j=1}^{n} x_j \geq \beta(U)$. *Then this inequality does not always define a facet of* C(A).

PROOF. Consider the 10×10 matrix A introduced in the

proof of Theorem 3 and add to it the row (0010111111). We call A' this 11×10 matrix and $B'=(V',U',E')$ its bipartite incidence graph.

First we show that *every* cutset of B' is critical. Using Theorem 5 and the fact that $\sum_{j=1}^{10} x_j \geq 3$ is a facet of C(A), we obtain that, in B', every cutset that contains the node u_{11} is critical. Now consider any cutset U_S of B', that does not contain u_{11}. W.l.o.g. assume that $v_{10} \in S$. Since $u_{11} \notin U_S$, we must also have $\{v_3,v_5,v_6,v_7,v_8,v_9\} \subseteq S$. Now, if $\{v_1,v_2\} \cap \bar{S} \neq \emptyset$, then $\{v_3,v_5\}$ covers $U \setminus U_S$ and the cutset is critical. So the last case to consider is when $\bar{S} = \{v_4\}$. Then $\{v_1,v_2\}$ covers $U \setminus U_S$ and again the cutset is critical. So all the cutsets of B' are critical.

Now we show that $\sum_{j=1}^{10} x_j \geq 3$ does not induce a facet of C(A'). Consider the 10 solutions introduced in the proof of Theorem 3 as satisfying $\sum_{j=1}^{10} x_j = 3$ for the problem C(A).

Note that one of these solutions does not satisfy the constraint generated by the new row 11 in matrix A'. In addition, no other feasible solution satisfies $\sum_{j=1}^{10} x_j = 3$. Since only nine affinely independent points satisfy this equation, the inequality $\sum_{j=1}^{10} x_j \geq 3$ does not induce a facet of C(A'). □

4. A NECESSARY AND SUFFICIENT CONDITION

Let A be a 0,1 matrix with column set V and row set U. In this section we consider general 0,1 inequalities,namely

inequalities of the form

$$\sum_{j \in S} x_j \geq \beta, \tag{5}$$

where $S \subseteq V$ and β is a positive integer. The main result, stated in Theorem 9, is a necessary and sufficient condition for such inequalities to induce facets of $C(A)$.

It will be convenient to assume in this section that A has at least two ones per row, thus guaranteeing that $C(A)$ is a polytope of full dimension.

Let D^S be the submatrix of A induced by a set S of columns of A and by the rows i such that $a_{ij} = 0$ for every $j \in \bar{S} = V \setminus S$. This operation is known as *deleting* the column set $\bar{S}$ from the 0,1 matrix A. Equivalently, consider the bipartite incidence graph $B = (V,U,E)$ associated with the matrix A and let $B(S)$ be the subgraph of B induced by the node set $S \cup R$ where $R \subseteq U$ contains the nodes that have no neighbor in $\bar{S}$. Then $B(S)$ is the bipartite incidence graph of D^S.

THEOREM 7. *If the inequality* (5) *defines a facet of* $C(A)$, *then* $\beta = \beta(D^S)$ *and* (5) *defines a facet of* $C(D^S)$.

PROOF. Consider a feasible solution such that $\sum_{j \in S} x_j = \beta(D^S)$ and $x_j = 1$ for every $j \in \bar{S}$. Since (5) is a valid inequality, this feasible solution implies $\beta(D^S) \geq \beta$. Now, if $\beta < \beta(D^S)$, then no feasible solution can satisfy (5) with equality, contradicting the fact that (5) is a facet. This shows $\beta = \beta(D^S)$.

To show that (5) defines a facet of $C(D^S)$, consider n affinely independent solutions $\{\tilde{x}^k\}_{1 \leq k \leq n}$ show-

ing that (5) is a facet of C(A). Project these n points in the space $x_j = 0$ for $j \in \bar{S}$. The n projections still satisfy (5) with equality. In addition, if the projected space had dimension less than $|S|-1$, then the space generated by $\{\tilde{x}^k\}_{1\leq k\leq n}$ would have dimension less than n-1. So the n projections define an affine space of dimension $|S|-1$. This shows that (5) defines a facet of $C(D^S)$ provided that this polyhedron has dimension $|S|$. This is indeed the case as a consequence of our assumption that A contains at least two ones per row and the definition of D^S. □

It is interesting to relate Theorem 7 to similar properties for the set packing problem. Given the 0,1 matrix A, denote by A^S the submatrix of A induced by a subset S of the columns of A and by G(S) the intersection graph associated with the matrix A^S. Consider the inequality

$$\sum_{j\in S} x_j \leq \alpha \,, \tag{6}$$

for some positive integer α. If the inequality (6) defines a facet of P(A), then $\alpha = \alpha(G(S))$ and (6) defines a facet of $P(A^S)$.

Unlike for the set packing problem, the matrix D^S needed in Theorem 7 does not usually contain all the rows of U. In fact, it is possible to have $\beta(D^S) < \beta(A^S)$, as shown by the following example.

$$A = \begin{array}{cc} & S \\ R & \left(\begin{array}{ccc|ccc} 1&1&0& & & \\ 0&1&1& &0& \\ 1&0&1& & & \\ \hline 1&0&0&1&1&0\\ 0&1&0&0&1&1\\ 0&0&1&1&0&1 \end{array}\right)\end{array}$$

The inequality $\sum_{j=1}^{3} x_j \geq 2$ defines a facet of C(A). The matrix D^S is the 3×3 submatrix of A induced by the row set R and the column set S. Note that $\beta(D^S) = 2$ and $\beta(A^S) = 3$.

For the set packing problem, Balas and Zemel (1977) gave a necessary and sufficient condition for a facet of $P(A^S)$ to also define a facet of P(A).

THEOREM 8 (Balas and Zemel 1977). *Let* A *be a* 0,1 *matrix and* A^S *the submatrix induced by a subset* S *of the columns of* A. *Assume that*

$$\sum_{j\in S} x_j \leq \alpha(G(S)) \qquad (7)$$

defines a facet of $P(A^S)$. *Then* (7) *defines a facet of* P(A) *if and only if, for every* $j \notin S$, *the cutset* $(S,\{j\})$ *of* $G(S \cup \{j\})$ *is not critical.*

Similalrly, the following theorem gives a necessary and sufficient condition for a facet of $C(D^S)$ to define a facet of C(A). Recall that B(S) denotes the bipartite incidence graph of D^S.

THEOREM 9. *Let* A *be a* 0,1 *matrix and* D^S *the submatrix*

of A *induced by a column set* S *and the rows* i *such that* $a_{ij} = 0$ *for every* $j \notin S$. *Assume that*

$$\sum_{j \in S} x_j \geq \beta(D^S) \tag{8}$$

defines a facet of $C(D^S)$. *Then* (8) *defines a facet of* $C(A)$ *if and only if, for every* $j \notin S$, *the cutset of* $B(S \cup \{j\})$ *induced by* $\{j\}$ *and* S *is* *not critical.*

PROOF. Assume that (8) defines a facet of $C(A)$ and consider $j \notin S$. There must exist a feasible solution such that $x_j = 0$ and $\sum_{i \in S} x_i = \beta(D^S)$, otherwise fewer than n affinely independent points would satisfy (8) with equality. This shows that it is possible to cover all the rows of $D^{S \cup \{j\}}$ with only $\beta(D^S)$ columns. In other words, the cutset of $B(S \cup \{j\})$ induced by $\{j\}$ and S is not a critical cutset.

Conversely, assume that, for every $j \notin S$, the cutset of $B(S \cup \{j\})$ induced by $\{j\}$ and S is not critical. This means that it is possible to cover all the rows of $D^{S \cup \{j\}}$ with a solution such that $x_j = 0$ and $\sum_{i \in S} x_i = \beta(D^S)$. To make this solution feasible for $C(A)$ it suffices to set $x_k = 1$ for every $k \notin S$ such that $k \neq j$. Denote by y^j such a solution to $C(A)$. Note that, if e^j denotes the unit vector such that $x_j = 1$, then $y^j + e^j$ is also a feasible solution and satisfies (8) with equality. Finally, let $\{x^k\}_{1 \leq k \leq |S|}$ be a set of $|S|$ affinely independent solutions such that $\sum_{i \in S} x_i^k = \beta(D^S)$ (such solutions exist since (8) defines a facet of $C(D^S)$), and set $x^k = 1$ for every $j \notin S$. We claim that the points

y^j, $y^j + e^j$ and x^k generate an affine space of dimension $n-1$. To see this, subtract y^j from $y^j + e^j$. The $n-|S|$ resulting unit vectors and the $|S|$ vectors x^k are linearly independent. This completes the proof. □

5. FACET-MINIMAL AND β-MAXIMAL 0,1 MATRICES

Given a 0,1 matrix A, we denote by L(A) the dimension of the affine space generated by the covers of A of cardinality β(A). We say that an m×n matrix A with 0,1 elements is *facet-minimal* if L(A) = n and, for every m×n matrix D with 0,1 elements such that D < A, then L(D) < L(A). This definition yields the following result.

PROPOSITION 1. *Let* A *be a* 0,1 *matrix. The inequality* $\sum_{j=1}^{n} x_j \geq \beta(A)$ *defines a facet of* C(A) *if and only if there exists a facet-minimal matrix* A_m *such that* $A_m \leq A$ *and* $\beta(A_m) = \beta(A)$.

We say that a 0,1 matrix A is *β-maximal* if, for every 0,1 matrix D of the same dimensions as A such that D > A, then β(D) < β(A). In other words, A is β-maximal if, turning into a 1 any entry of A which is currently 0 decreases the covering number.

PROPOSITION 2. *Let* A *be a* 0,1 *matrix and let* A_M *be a β-maximal matrix such that* $A_M \geq A$ *and* $\beta(A_M) = \beta(A)$. *If the inequality* $\sum_{j=1}^{n} x_j \geq \beta(A)$ *induces a facet of* C(A), *then it also induces a facet of* C(D) *for every* 0,1 *matrix* D *such that* $A \leq D \leq A_M$.

PROOF. The same n affinely independent solutions that show that $\sum_{j=1}^{n} x_j \geq \beta(A)$ defines a facet of C(A) can also be used for C(D). □

PROPOSITION 3. *Let* A *be a* β-*maximal* 0,1 *matrix containing at least one zero per row and let* B = (V,U,E) *be the bipartite incidence graph of* A. *Let* $G^* = (V,E^*)$ *the critical graph associated to* B. *The inequality*

$$\sum_{j=1}^{n} x_j \geq \beta(A)$$

defines a facet of the set covering polytope C(A) *if and only if the graph* $G^* = (V,E^*)$ *is connected.*

PROOF. The fact that the condition is sufficient is stated in Theorem 2. So we have to prove that the condition is necessary.

Consider two nonorthogonal columns j and k. Let $u_i \in U$ be one of the common neighbors of v_j and v_k. Let $a_{i\ell}$ be a zero element of row i. Turning this element into a 1 would decrease the covering number of A. Thus, we also have $\beta(U \setminus \{u_i\}) < \beta(U)$. This shows that $(v_j,v_k) \in E^*$.

To prove that $G^* = (V,E^*)$ is connected, it suffices to note that B is connected (otherwise $\sum_{j=1}^{n} x_j \geq \beta(A)$ would be the sum of at least two valid inequalities and, therefore, would not induce a facet). □

It may be interesting to note that the concept of β-maximal matrix is also a central notion for the facial description of the simple plant location polytope, These matrices were introduced by Cho, Padberg and Rao (1983)

under the name of maximal pd-adjacency matrices.

The simple plant location polytope is the convex full of the solutions to

$$\begin{cases} \sum_{j=1}^{n} y_{ij} = 1 & \text{for } i = 1,\dots,m \\ y_{ij} + \bar{x}_j \le 1 & \text{for } i=1,\dots,m \text{ and } j=1,\dots,n \\ y_{ij}\ ,\ \bar{x}_j \in \{0,1\}. \end{cases}$$

Let G be the intersection graph associated with this 0,1 constraint matrix. Necessary and sufficient conditions for a 0,1 inequality to define a facet of the simple plant location polytope were given by Cho, Padberg and Rao (1983) and Cornuejols and Thizy (1982). Let $I \subseteq \{1,\dots,m\}$, $J \subseteq \{1,\dots,n\}$ and let A be a 0,1 matrix with rows indexed by I and columns indexed by J. The inequality

$$\sum_{i\in I} \sum_{j\in J} a_{ij}y_{ij} + \sum_{j\in J} \bar{x}_j \le \alpha \qquad (9)$$

induces a facet of the simple plant location polytope if and only if

(i) the matrix A is β-maximal and has at least two ones and one zero per row, where $\beta = |I|+|J|-\alpha$;

(ii) α is the maximum size of a node packing in the subgraph of H induced by the nodes that have a positive coefficient in (9).

It was shown that A is β-maximal if and only if, in the graph $K = (V,E)$ defined above, the subgraph $K^* = (V,E^*)$ induced by the critical edges of K is connected. (Here, the notion of critical edge refers to node packing).

6. ODD HOLES AND THE ASSOCIATED β-MAXIMAL MATRICES

The 0,1 matrices of odd order with exactly two ones per row and per column play a central role in the polyhedral theory of packings and coverings. We call such matrices *odd holes*.

Let A be a 0,1 matrix. If A does not contain an odd hole as a submatrix, then it is well known that $C(A) = \{x\colon Ax \geq 1,\ 0 \leq x \leq 1\}$, in other words no additional facet is required in the description of C(A). See Fulkerson, Hoffman and Oppenheim (1974). On the other hand, consider the case where A is an odd hole of order n. Then the inequality

$$\sum_{j=1}^{n} x_j \geq \frac{n+1}{2} \tag{10}$$

defines a facet of C(A). In fact, the same statement remains true for any matrix A satisfying $H \leq A \leq H^*$ where H is an odd hole and H^* is an $\frac{n+1}{2}$ - maximal matrix with the property that $H^* \geq H$. Note that H is facet-minimal.

In this section we characterize all such matrices H^*. From an algorithmic point of view, given a square 0,1 matrix A of odd order and an odd hole $H \leq A$, our characterization allows us to decide in polynomial time whether the inequality (10) is valid and yields a facet of C(A). Of course, if only A is given, finding H is NP-hard in general as it amounts to finding a Hamilton cycle in a bipartite graph. However, for the simple plant location polytope mentioned in the previous section, odd holes can be generated from any odd sets I and J such that $|I| = |J|$. Then our characterization

provides a polynomial algorithm for performing sequential liftings of the y_{ij} variables, $i \in I$ and $j \in J$. See Padberg (1973) for an introduction to sequential variable liftings. **Cornuejols and Thizy (1982) gave all the $\frac{n+1}{2}$ - maximal matrices H^* for $n = 3, 5$ and 7. They were derived using a facial description of the convex hull of the odd hole liftings in the simple plant location polytope, see Thizy (1981).**

As in earlier sections, we find it useful to work on the bipartite incidence graph $B = (V,U,E)$ of the 0,1 matrix A. We assume that $|V| = |U| = n$ and that we have a subset $E_H \subseteq E$ with $|E_H| = 2n$ such that the graph $H = (V,U,E_H)$ is a cycle. We call the set of edges $E_C = E \setminus E_H$ the *chord set* of H and each edge $e = (v_i,u_j) \in E_C$ a *chord* of H. The graph B is called a *chorded cycle* spanned by H.

In the remainder of this section we assume that $n \geq 3$ is an odd integer. We say that H is an odd cycle. More generally, we say that a cycle of the bipartite graph B is *odd* if its number of edges is 2k where k is odd.

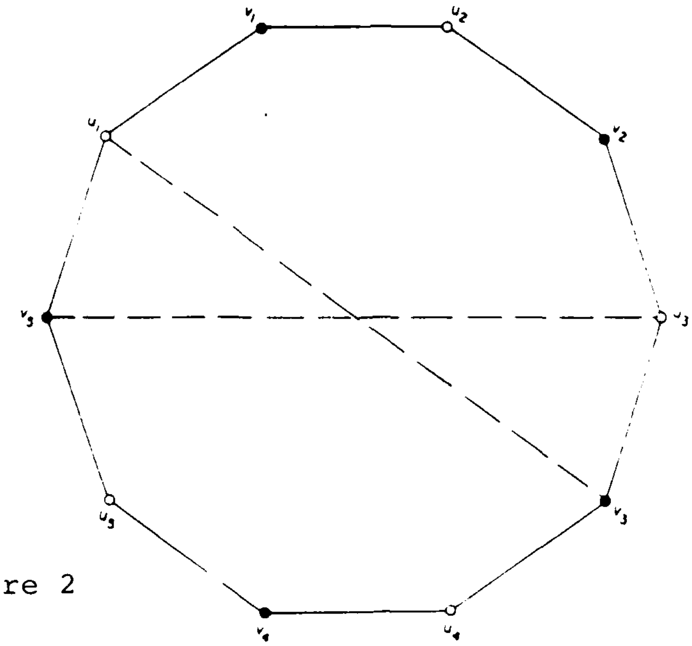
We say that a chord H is *odd* if it induces two odd cycles in H. In other words, the chord $e_{ij} = (v_i,u_j)$ is odd if each of the two paths of H joining v_i to u_j forms an odd cycle with e_{ij}. If a chord is not odd, it is said to be even.

Each chord $e_{ij} = (v_i,u_j)$ induces a partition of the nodes of V into an even set $E_{ij}(v_i)$ and an odd set $O_{ij}(v_i)$ defined as follows. Consider the graph obtained from H by removing u_j and the two incident edges. Given $v_h \in V$, let d_{ih} be the length (number of edges) of the unique path from v_i to v_h in this graph. We define $E_{ij}(v_i) = \{v_h \in V : d_{ih} \equiv 0 \pmod 4\}$ and $O_{ij}(v_i) = \{v_h \in V : d_{ih} \equiv 2 \pmod 4\}$.

Note that $|E_{ij}(v_i)| = |O_{ij}(v_i)|+1$ when the chord e_{ij} is odd and $|E_{ij}(v_i)| = |O_{ij}(v_i)|-1$ when it is even.

Two chords $e_{ij} = (v_i,u_j)$ and $e_{hk} = (v_h,u_k)$ are said to be *crossing* if their nodes v_i,v_h,u_j,u_k appear in this order on the graph H. The two chords are said to be *compatible* if $v_h \in E_{ij}(v_i)$. Equivalently $v_i \in E_{hk}(v_h)$.

EXAMPLE



$E_{31}(v_3) = \{v_1,v_3,v_5\}$

$O_{31}(v_3) = \{v_2,v_4\}$

Figure 2

The graph of Figure 2 is a chorded cycle spanned by the 10-cycle H drawn with solid edges. The chords e_{31} and e_{53} are both odd. They are crossing as v_3,v_5,u_1,u_3 appear in this order on H. Finally they are compatible as $v_5 \in E_{31}(v_3)$.

The chord set E_C of a chorded cycle $B = (V,U,E_H \cup E_C)$ is said to be *compatible* if

(i) every chord is odd, and

(ii) every pair of crossing chords is compatible.

THEOREM 10. *Let* A *be a* 0,1 *matrix and let* $B=(V,U,E)$ *be its incidence bipartite graph. Assume that* $|V|=|U|=n$

is odd and that there exists $E_H \subseteq E$ *with* $|E_H| = 2n$ *such that* $H = (V,U,E_H)$ *is a cycle. Let* $E_C = E \setminus E_H$ *be the set of chords of* H. *Then the inequality*

$$\sum_{j=1}^{n} x_j \geq \frac{n+1}{2} \tag{11}$$

defines a facet of C(A) *if and only if the chord set* E_C *is compatible.*

Before we prove the theorem, we note a property of pairs of odd crossing chords. Given a chord $e_{ij}=(v_i,v_j)$, consider a partition of the nodes of U into an even set and an odd set similar to what we did for the set V. In the graph obtained from H by removing v_i and the two incident edges, we define d_{jk} to be the length of the unique path joining u_j to any other node $u_k \in U$. Then we define

$E_{ij}(u_j) = \{u_k \in U : d_{jk} \equiv 0 \pmod 4\}$ and

$O_{ij}(u_j) = \{u_k \in U : d_{jk} \equiv 2 \pmod 4\}$.

Now assume that $e_{ij} = (v_i,u_j)$ and $e_{hk} = (v_h,u_k)$ are two odd crossing chords. We claim that

$$v_h \in E_{ij}(v_i) \textit{ if and only if } u_k \in E_{ij}(u_j) \tag{12}$$

To see this, remember that, since e_{ij} and e_{hk} are crossing, the nodes v_i, v_h, u_j and u_k appear in this order on the cycle H. So, the edges of H can be partitioned into four paths joining v_i to v_h, v_h to u_j, u_j to u_k and

u_k to v_i. Let d_{ih}, d_{hj}, d_{jk} and d_{ki} be the respective lengths of these paths. Since e_{ij} and e_{hk} are odd chords, we have

$$d_{ih} + d_{hj} + 1 = 2w, \text{ where } w \text{ is an odd integer,}$$

and $d_{hj} + d_{jk} + 1 = 2z$, where z is odd.

So $d_{ih} = d_{jk} + 2(w-z)$, i.e.

$$d_{ih} \equiv d_{jk} \pmod 4.$$

This proves the claim (12).

PROOF OF THEOREM 10. Assume that the chord set E_C is not compatible. First assume that (i) does not hold, say $e_{ij} = (v_i, u_j)$ is an even chord. Then $E_{ij}(v_i)$ is a cover of cardinality $\frac{n-1}{2}$, contradicting the validity of inequality (11). Now assume that every chord is odd but that (ii) does not hold for some pair of crossing chords $e_{ij} = (v_i, u_j)$, $e_{hk} = (v_h, u_k)$. Let V_1 and U_1 be the nodes of V and U respectively in the path from v_i to u_j which contains u_k. Similarly let V_2 and U_2 be the nodes of V and U in the path from v_i to u_j which contains v_h. Note that $v_i \in V_1$ and V_2, and $u_j \in U_1$ and U_2. Consider the path $P = (u_k, v_\ell, u_m, \ldots, v_i)$ from u_k to v_i which does not contain u_j and v_h (see Figure 3.) Define D_1 to be the minimum cover of U_1 by nodes of V_1 that contains the two nodes of V_1 adjacent to the node u_m on P. Since the two chords e_{ij} and e_{hk} are not compatible, $v_h \in O_{ij}(v_i)$. Because the chord e_{hk} is odd, this implies that $v_\ell \in O_{ij}(v_i)$. Since we have chosen D_1 so that $v_\ell \in D_1$, we have $v_i \in D_1$. Now define D_2 to be a minimum cover of U_2 by nodes of V_2

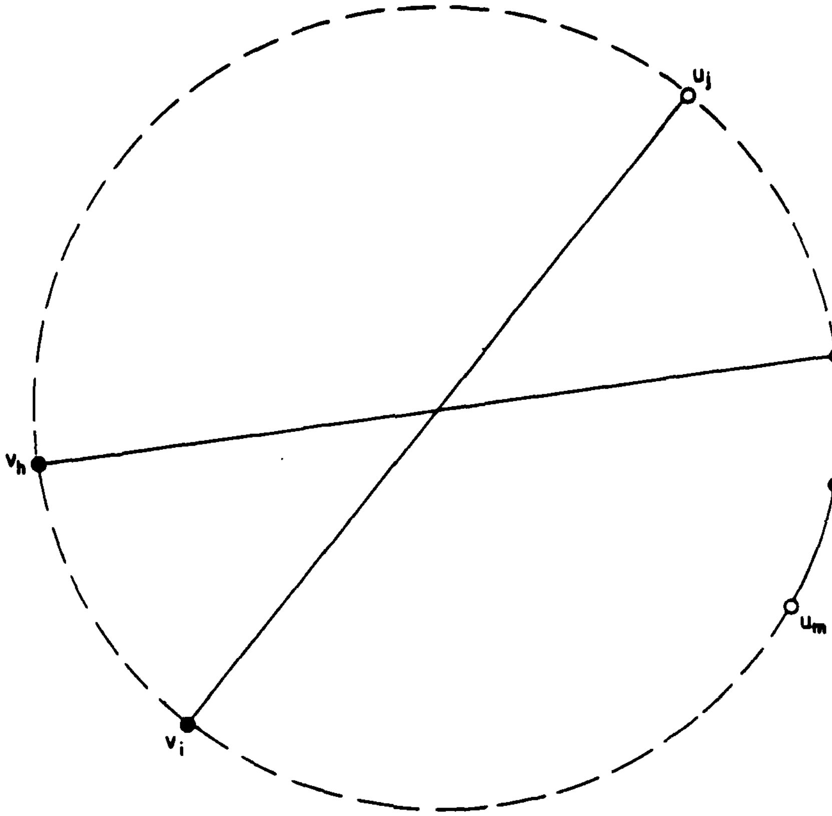


Figure 3.

that contains both v_i and v_h. The chord e_{ij} being odd, we have $|D_1| = (|V_1|+1)/2$ and $|D_2| = (|V_2|+1)/2$. Now define D as $D = D_1 \cup D_2 \setminus \{v_\ell\}$. As u_m and u_k are covered by nodes of D, it is clear that D is a cover of U. In addition $|D| = |D_1| + |D_2|-2$ as v_ℓ has been removed and v_i appears in both D_1 and D_2. So $|D| = (|V_1| + |V_2|-2)/2$. Since $|V_1| + |V_2| = n+1$, we have $|D| = (n-1)/2$ and therefore the inequality (11) is not valid, a contradiction. So the compatibility of the chord set is a necessary condition for (11) to be a facet.

Now we show that it is also sufficient. We only have to prove that (11) is valid, i.e. we want to show that,if D is a minimum cover, then $|D| = (n+1)/2$. Obviously, D is still a cover in the graph obtained from B by removing the set of chords $E_1 = \{(v,u) \in E : v \notin D\}$. So in the remainder we assume that we work in such a reduced graph.

Let $e_{ij} = (v_i, u_j)$ be a chord such that the subgraph induced by one of the two paths of H joining v_i to u_j, say P, is a chordless cycle.

Consider all chords $e_{\ell k} = (v_\ell, u_k)$ such that $u_k \in P$. Note that, since e_{ij} and $e_{\ell k}$ are crossing chords, $u_k \in E_{ij}(u_j)$ as a consequence of claim (12). In other words, no node of $P \cap O_{ij}(u_j)$ is covered by a node not in P.

Now consider all chords $e_{\ell k} = (v_\ell, u_k)$ such that $v_\ell \in P$. Since e_{ij} and $e_{\ell k}$ are crossing, $v_\ell \in P \cap E_{ij}(v_i)$.

It follows from these two observations that we can always find a minimum cover which coincides in P with $E_{ij}(v_i)$. So let us assume that D has this property. Consequently D not only covers $P \cap O_{ij}(u_j)$ but also $P \cap E_{ij}(u_j)$. So the chords (v_ℓ, u_k) where $u_k \in P$ can be removed while leaving D a feasible cover. Let E_2 be this set of chords.

Now we will show the result by induction on n. Namely we assume that the inequality (11) is valid for chorded cycles with compatible chord sets having less than 2n nodes and we prove that it is also valid for any chorded cycle with compatible chord set having 2n nodes. To this end, construct a new graph $B'=(V',U',E')$ from B in the following way. Remove from V and U the nodes of P except v_i and u_j, remove from the set E all the edges of P and the chord sets E_1 and E_2 defined above. In addition, for each chord (v_ℓ, u_k) where $v_\ell \in P \setminus \{v_i\}$, introduce a new chord (v_i, u_k). Let $D' = D \cap V'$. By construction, D' is a cover of U' and B' is a chorded cycle with compatible chord set. Denote by q the number of nodes of $P \cap (V \setminus \{v_i\})$. Because e_{ij} is an odd chord, then q is even. Moreover, $|V'| = n - q$ and $|D'| = |D| - q/2$. By the induction hypothesis $|D'| \geq \frac{(n-q)+1}{2}$. Consequently $|D| \geq \frac{n+1}{2}$. □

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In this paper, we consider inequalities of the form $\Sigma\, \alpha_j x_j \geq \beta$, where α_j equals 0 or 1, and β is a positive integer. We give necessary and sufficient conditions for such inequalities to define facets of the set covering polytope associated to a 0,1 constraint matrix A. These conditions are in terms of critical edges and critical cutsets defined in the bipartite incidence graph associated to A, andare very much in the spirit of the work of Balas and Zemel on the set packing problem where similar notions were defined in the intersection graph of A. Furthermore, we give a polynomial characterization of a

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class of 0,1 facets defined from chorded cycles induced in the bipartite incidence graph. This characerization also yields all the 0,1 liftings of odd-hole inequalities for the simple plant location polytope.

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